QUADRATIC HARMONIC MORPHISMS AND O-SYSTEMS

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Abstract

We introduce O-systems (Definition 3.1) of orthogonal transformations of $\mathbb{R}^m$, and establish $1-1$ correspondences both between equivalence classes of Clifford systems and that of O-systems, and between O-systems and orthogonal multiplications of the form $\mu : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$, which allow us to solve the existence problems both for O-systems and for umbilical quadratic harmonic morphisms (Theorems 4.4 and 4.5) simultaneously. The existence problem for general quadratic harmonic morphisms is then solved (Theorem 4.9) by the Splitting Lemma (Lemma 4.7). We also study properties (see, e.g., Theorems 5.2 and 5.11) possessed by all quadratic harmonic morphisms for fixed pairs of domain and range spaces (§5).

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1. Introduction

A map $\varphi : (M^m, g) \longrightarrow (N^n, h)$ between two Riemannian manifolds is called a **harmonic map** if the divergence of its differential vanishes. Such maps are the extrema of the energy functional $\frac{1}{2}\int_D |d\varphi|^2$ over compact domain D in M. For a detailed account on harmonic maps we refer to [10, 11, 12] and the references therein. **Harmonic morphisms** are a special subclass of harmonic maps which preserve solutions of Laplace's equation in the sense that for any harmonic function $f : U \longrightarrow \mathbb{R}$, defined on an open subset U of N with $\varphi^{-1}(U)$ non-empty, $f \circ \varphi : \varphi^{-1}(U) \longrightarrow \mathbb{R}$ is a harmonic function. In other words, φ pulls back germs of harmonic functions on N to germs of harmonic functions on M. In the theory of stochastic process, harmonic morphisms $\varphi : (M, g) \longrightarrow (N, h)$ are found to be Brownian pat! h preserving mappings meaning that they map Brownian motions on M to Brownian motions on N([6, 24]). It is well-known (see [16, 23]) that a map between Riemannian manifolds is a harmonic morphism if and only if it is both a harmonic map and a horizontally weakly conformal map. For a map $\varphi : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ between Euclidean spaces, with $\varphi(x) = (\varphi^1(x), ..., \varphi^n(x))$, the harmonicity and horizontally weakly conformality are equivalent to the following conditions respectively:

$$\sum_{i=1}^{m} \frac{\partial^2 \varphi^\alpha}{\partial x_i^2} = 0 \tag{1}$$

$$\sum_{i=1}^{m} \frac{\partial \varphi^\alpha}{\partial x_i}\frac{\partial \varphi^\beta}{\partial x_i} = \lambda^2(x)\delta^{\alpha\beta}\ ,\ \ \alpha, \beta = 1, 2, ..., n. \tag{2}$$

where $(x_1, \ldots, x_m)$ are the standard coordinates of $\mathbb{R}^m$.

In recent years, much work has been done in classifying and constructing harmonic morphisms from certain model spaces to other model spaces (see e.g. [1, 2, 3, 4, 5, 17, 18, 19, 20, 37, 38, 30, 27, 29]). Concerning harmonic morphisms between Euclidean spaces, Baird [1] has studied harmonic morphisms $\varphi : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ defined by homogeneous polynomials of degree p. He has obtained a necessary condition on

the dimensions of the domain and the range spaces for such harmonic morphisms to exist; he also gives a possible way to construct such harmonic morphisms from a single polynomial. For quadratic harmonic morphisms, i.e., harmonic morphisms defined by homogeneous polynomials of degree 2, he proves (Theorem 7.2.7 in [1]) that an orthogonal multiplication $\mu : \mathbb{R}^p \times \mathbb{R}^q \longrightarrow \mathbb{R}^n$ is a harmonic morphism if and only if the dimensions $p = q = n$ and $n = 1, 2, 4$, or 8. It is well-known that the standar! d multiplications of the real algebras of real, complex, quaternionic and Cayley numbers are both orthogonal multiplications and harmonic morphisms. Baird also shows that any Clifford system, i.e., an n-tuple $(P_1, ..., P_n)$ of symmetric endomorphisms of $\mathbb{R}^{2m}$ satisfying $P_iP_j + P_jP_i = 2\delta_{ij}Id$ for $i, j = 1, ...n$, defines a quadratic harmonic morphism $\varphi : \mathbb{R}^{2m} \longrightarrow \mathbb{R}^n$ by

$$\varphi(X) = (\langle P_1X, X\rangle, \dots , \langle P_nX, X\rangle). \tag{3}$$

The author proves [27] that the "complete lift" (Definition 2.1 in [27]) of any quadratic harmonic morphism $\varphi : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is again a quadratic harmonic morphism $\overline{\varphi} : \mathbb{R}^{2m} \longrightarrow \mathbb{R}^n$, thus provides a method of constructing quadratic harmonic morphisms from given ones. In [30], a classification of quadratic harmonic morphisms is given and it is also shown that any umbilical quadratic harmonic morphism arises from a Clifford system.

In this work, we solve completely the existence and the classification problem for quadratic harmonic morphisms by introducing O-systems of orthogonal transformations of the domain space. In §2 we recall the first classification theorem obtained in [30], discuss a direct sum construction of harmonic morphisms and establish a $1-1$ correspondence between equivalence classes of Clifford systems and umbilical quadratic harmonic morphisms (Theorem 2.6). In §3 we introduce the notion of O-systems and obtain a $1-1$ correspondence between equivalence classes of Clifford systems and that of O-systems (Theorem 3.5);

In §4 we establish a $1-1$ correspondence between O-systems and orthogonal multiplications (Theorem 4.2). Putting these together and using the Splitting Lemma (Lemma 4.7), we obtain our main theorems (Theorems 4.5 and 4.9). §5 is devoted to study properties possessed by all quadratic harmonic morphisms for fixed pairs of domain and range spaces (including quadratic harmonic morphisms into $\mathbb{R}^{2!}$, $\mathbb{R}^3$, $\mathbb{R}^4$, $\mathbb{R}^5$, $\mathbb{R}^8$, and $\mathbb{R}^9$). Among other things, we show that any quadratic harmonic morphism arises from a single quadratic function (Theorem 5.2). Also we show that we can generalise the Hopf construction to "domain-minimal" but not "range-maximal" quadratic harmonic morphisms (Theorem 5.4) and finally quadratic harmonic morphisms in dimensions 2n to n or $n+1$ ($n = 1, 2, 4, 8$) are shown to be equivalent to standard maps.

Applications of O-systems in classifying orthogonal multiplications $F(n, m; m)$ and in constructing isoparametric functions on, and minimal submanifolds of spheres will appear in the author's paper [28].

2. Quadratic harmonic morphisms and Clifford systems

We use $O(\mathbb{R}^m)$ and $S(\mathbb{R}^m)$ to denote the set of all orthogonal endomorphisms and that of all symmetric endomorphisms of $\mathbb{R}^m$ respectively. When the latter is viewed as a Euclidean space it is understood to have the inner product defined by

$$\langle A, B\rangle = \frac{1}{m} tr(AB) \tag{4}$$

for any $A, B \in S(\mathbb{R}^m)$.

A **quadratic harmonic morphism** $\varphi : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is a harmonic morphism whose components are quadratic functions (i.e. homogeneous polynomials of degree 2) in $x_1, \ldots, x_m$. We use $H_2(m, n)$ to denote the set of all quadratic harmonic morphisms $\varphi : \mathbb{R}^m \longrightarrow \mathbb{R}^n$. Let $\varphi \in H_2(m, n)$ with $\varphi(X) = (X^t A_1 X, \ldots, X^t A_n X)$. We have proved ([30]) that the component matrices A_i have the same rank which

is always an even number which we call the Q-rank of φ. The quadratic harmonic morphism φ is said to be **Q-nonsingular** if Q-rank(φ) equals the domain dimension, otherwise φ is said to be **Q-singular**; We also prove that the A_i have the same spectrum which consists of pairs $\pm\lambda$ of eigenvalues and possibly 0. Then we obtained the following **Classification Theorem** [30]**:** Let $\varphi : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ $(m \geq n)$ be a quadratic harmonic morphism.

(I) If φ is Q-nonsingular, then $m = 2k$ for some $k \in \mathbb{N}$ and, with respect to suitable orthogonal coordinates in $\mathbb{R}^m$, φ assumes the normal form

$$\begin{aligned}(5)\qquad \varphi(X) = \Bigg(& X^t \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix} X,\ X^t \begin{pmatrix} 0 & B_1 \\ B_1^t & 0 \end{pmatrix} X, \dots, \\ & X^t \begin{pmatrix} 0 & B_{n-1} \\ B_{n-1}^t & 0 \end{pmatrix} X \Bigg)\end{aligned}$$

where $B_i,\ D \in GL(\mathbb{R}, k)$ with D diagonal having the positive eigenvalues as diagonal entries, and satisfy

$$(6)\qquad \begin{cases} DB_i = B_i D \\ B_i^t B_i = D^2 \\ B_i^t B_j = -B_j^t B_i.\ \ (i, j, = i, ..., n-1, i \neq j). \end{cases}$$

(II) Otherwise Q-rank$(\varphi) = 2k$ for some $k,\ 0 \leq k < m/2$, and φ is the composition of an orthogonal projection $\pi : \mathbb{R}^m \longrightarrow \mathbb{R}^{2k}$ followed by a Q-nonsingular quadratic harmonic morphism $\varphi_1 : \mathbb{R}^{2k} \longrightarrow \mathbb{R}^n$.

It follows that any quadratic harmonic morphism from an odd-dimensional space is the composition of an orthogonal projection followed by a Q-nonsingular quadratic harmonic morphism from an even-dimensional space. Thus to study quadratic harmonic morphisms it suffices to consider Q-nonsingular ones from even-dimensional spaces.

Definition 2.1. *Let (M, g) and (N, h) be two Riemannian manifolds. Suppose that $\varphi : M \longrightarrow \mathbb{R}^n$ and $\tilde{\varphi} : N \longrightarrow \mathbb{R}^n$ are two C^∞ maps. Then*

the **direct sum** *of* φ *and* $\tilde{\varphi}$ *is a map*

$$\varphi \oplus \tilde{\varphi} : M \times N \longrightarrow \mathbb{R}^n$$

defined by

$$(\varphi \oplus \tilde{\varphi})(p, q) = \varphi(p) + \tilde{\varphi}(q)$$

where $M \times N$ is the product of M and N, endowed with the Riemannian product metric $G = (g, h)$.

For more results on the direct sum construction of harmonic morphisms see [29].

Remark 2.2. *It follows from Ou* [29] *that the direct sum of any two harmonic morphisms is again a harmonic morphism. In particular, the direct sum of two quadratic harmonic morphisms $\varphi : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ with $\varphi(X) = (X^t A_1 X, \ldots, X^t A_n X)$ and $\tilde{\varphi} : \mathbb{R}^l \longrightarrow \mathbb{R}^n$ with $\tilde{\varphi}(X) = (X^t B_1 X, \ldots, X^t B_n X)$ is a quadratic harmonic morphism $\varphi \oplus \tilde{\varphi} : \mathbb{R}^{m+l} \longrightarrow \mathbb{R}^n$ given by*

$$(\varphi \oplus \tilde{\varphi})(X, Y) = (X^t A_1 X + Y^t B_1 Y, \ldots, X^t A_n X + Y^t B_n Y)$$
$$= \left((X^t\ Y^t) \begin{pmatrix} A_1 & 0 \\ 0 & B_1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}, \ldots, (X^t\ Y^t) \begin{pmatrix} A_n & 0 \\ 0 & B_n \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \right)$$

A quadratic harmonic morphism is said to be **separable** if it can be written as the direct sum of two quadratic harmonic morphisms from smaller dimensional domain spaces. Concerning the existence of quadratic harmonic morphisms we note that if there exists $\varphi \in H_2(m, n)$ then, by projection, $H_2(m, k) \neq \emptyset$ for $1 \leq k < n$; On the other hand, by using the direct sum construction, we know that if $H_2(m, n) \neq \emptyset$ then $H_2(km, n) \neq \emptyset$ for $k \in \mathbb{N}$.

Definition 2.3. *i) A quadratic harmonic morphism $\varphi \in H_2(m, n)$ is said to be* **range-maximal** *if for fixed m, n is the largest range dimension such that $H_2(m, n) \neq \emptyset$; It is said to be* **domain-minimal** *if for fixed n, m is the smallest domain dimension such that $H_2(m, n) \neq \emptyset$.*
ii) Two quadratic harmonic morphisms $\varphi, \tilde{\varphi} \in H_2(m, n)$ are said to be **domain-equivalent**, *denoted by $\varphi \overset{d}{\sim} \tilde{\varphi}$, if there exists an isometry*

$G \in O(\mathbb{R}^m)$ such that $\varphi = \tilde{\varphi} \circ G$. They are said to be **bi-equivalent**, *denoted by $\varphi \overset{bi}{\sim} \tilde{\varphi}$, if there exist isometries $G \in O(\mathbb{R}^m), H \in O(\mathbb{R}^n)$ such that $\varphi = H^{-1} \circ \tilde{\varphi} \circ G$.*

Obviously, a quadratic harmonic morphism is domain-minimal if and only if it is not separable. Also, a domain-minimal quadratic harmonic morphism is always Q-nonsingular.

Definition 2.4. *(see, e.g.* [15]*)*

i) A $(2m, n)$-dimensional **Clifford system** *is an n-tuple $(P_1, \dots, P_n)$, denoted by $\{P_i\}$ for short, of symmetric endomorphisms of $\mathbb{R}^{2m}$ satisfying*

$$P_iP_j + P_jP_i = 2\delta_{ij} Id \ (i, j = 1, ..., n). \tag{7}$$

The set of all $(2m, n)$-dimensional Clifford systems is denoted by $C(2m, n)$.
ii) A **representation** *of a Clifford system $\{P_i\} \in C(2m, n)$ is a n-tuple $(A_1, \dots, A_n)$ of symmetric matrices such that, with respect to some orthonormal basis in $\mathbb{R}^{2m}$, A_i is the representation of P_i for $i = 1, ..., n$ respectively. A Clifford system is sometimes specified by its representation as $\{A_i\}$.*
iii) Let $\{P_i\} \in C(2m, n)$ and $\{Q_i\} \in C(2l, n)$, then $\{P_i \oplus Q_i\}$ is a Clifford system on $\mathbb{R}^{2m+2l}$, the so-called **direct sum** *of $\{P_i\}$ and $\{Q_i\}$.*
iv) A Clifford system $\{P_i\} \in C(2m, n)$ is said to be **irreducible** *if it is not possible to write $\mathbb{R}^{2m}$ as a direct sum of two non-trivial subspaces which are invariant under all P_i.*
v) Two Clifford systems $\{P_i\}$, $\{Q_i\} \in C(2m, n)$ are said to be **algebraically equivalent**, *denoted by $\{P_i\} \overset{a}{\sim} \{Q_i\}$, if there exists $A \in O(\mathbb{R}^{2m})$ such that $Q_i = AP_iA^t$ for all $i = 1, \dots, n$. They are said to be* **geometrically equivalent**, *denoted by $\{P_i\} \overset{g}{\sim} \{Q_i\}$, if there exists $B \in O(span\{P_1, \dots, P_n\}) \subset O(S(\mathbb{R}^{2m}))$ such that $\{B(P_i)\}$ and $\{Q_i\}$ are algebraically equivalent.*

A quadratic harmonic morphism is said to be **umbilical** (see [30]) if all the positive eigenvalues of one (and hence all) of its component

matrices are equal.

It follows form Baird [1] that any Clifford system $\{P_i\} \in C(2m, n+1)$ gives rise, by (3), to an umbilical quadratic harmonic morphism with positive eigenvalue 1. On the other hand, we know from [30] that, up to a constant factor, any umbilical quadratic harmonic morphism $\varphi \in H_2(2m, n+1)$ is domain-equivalent to one arising from the Clifford system

$$\left(\begin{pmatrix} I_m & 0 \\ 0 & -I_m \end{pmatrix}, \begin{pmatrix} 0 & \tau_1 \\ \tau_1^t & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & \tau_n \\ \tau_n^t & 0 \end{pmatrix} \right) \tag{8}$$

where $\tau_i \in O(\mathbb{R}^m)$, and satisfy

$$\tau_i^t \tau_j + \tau_j^t \tau_i = 2\delta_{ij} Id \; (i, j = 1, ..., n). \tag{9}$$

Remark 2.5. *Note that an umbilical quadratic harmonic morphism with positive eigenvalue λ can always be normalized, by a change of the scalar, to be the one with positive eigenvalue* 1*. Thus the study of umbilical quadratic harmonic morphisms reduces to the study of umbilical quadratic harmonic morphisms with positive eigenvalue* 1*.*

Let $H_2^1(2m, n)$ denote the subset of $H_2(2m, n)$ consisting of all umbilical quadratic harmonic morphisms with positive eigenvalue 1. Then, as we have seen from the above, we have a map

$$F : C(2m, n) \longrightarrow H_2^1(2m, n). \tag{10}$$

with $F(\{P_i\})$ defined by (3).

Theorem 2.6. *Let F be the map defined by (10). Then*
(i) $\{P_i\} \overset{a}{\sim} \{Q_i\}$ if and only if $F(\{P_i\}) \overset{d}{\sim} F(\{Q_i\})$;
(ii) $\{P_i\} \overset{g}{\sim} \{Q_i\}$ if and only if $F(\{P_i\}) \overset{bi}{\sim} F(\{Q_i\})$.
(iii) F preserves the direct sum operations in the following sense:

$$F(\{Q_i \oplus R_i\}) \overset{d}{\sim} F(\{Q_i\}) \oplus F(\{R_i\}), \tag{11}$$

and hence F preserves reducibility in the sense that $\{P_i\}$ is irreducible if and only if $F(\{P_i\})$ is domain-equivalent to an umbilical quadratic harmonic morphism which is not separable.

Proof. (i) is obviously true. For (ii) we first note that when $S(\mathbb{R}^m)$ is viewed as a Euclidean space with the inner product defined by Equation (4), then any $\{P_i\} \in C(2m,n)$ becomes an orthonormal set. Thus $\mathrm{span}\{P_1,\dots,P_n\}$ can be identified with $\mathbb{R}^n$ provided with the standard inner product, and hence $O(span\{P_1,\dots,P_n\}) \cong O(\mathbb{R}^n)$. Now let $B(P_i) = a_i^j P_j$. Then it is easy to see that $B \in O(span\{P_1,\dots,P_n\})$ if and only if $(a_i^j) \in O(\mathbb{R}^n)$. It is routine to check that $\{B(P_i)\}$ is indeed a Clifford system $\in C(2m,n)$. By definition, we have

$$\begin{aligned}F\left(\{B(P_i)\}\right)(X) &= (\langle B(P_1)X, X\rangle, \dots, \langle B(P_n)X, X\rangle)\\ &= (a_1^j\langle P_jX, X\rangle, \dots, a_n^j\langle P_jX, X\rangle)\\ &= H^{-1} \circ F\left(\{P_i\}\right)(X).\end{aligned}$$

where $H = (a_i^j)^{-1} \in O(\mathbb{R}^n)$. By Definition 2.4, $\{P_i\} \overset{g}{\sim} \{Q_i\}$ if and only if $\{B(P_i)\} \overset{a}{\sim} \{Q_i\}$, which, by (i), is equivalent to $F\left(\{B(P_i)\}\right) \overset{d}{\sim} F\left(\{Q_i\}\right)$, i.e., $F\left(\{Q_i\}\right) = H^{-1} \circ F\left(\{P_i\}\right) \circ G$ for some $G \in O(\mathbb{R}^m)$, which means exactly that $F\left(\{P_i\}\right)$ and $F\left(\{Q_i\}\right)$ are bi-equivalent. This ends the proof of (ii).

Now we prove (iii), by using 1) of Lemma 3.4. it is easy to check that Equation (11) holds for any $\{Q_i\} \in C(2k,n)$ and $\{R_i\} \in C(2l,n)$. Now suppose that $\{P_i\} \in C(2m,n)$ is irreducible, we are to prove that $F\left(\{P_i\}\right)$ is domain-equivalent to an umbilical quadratic harmonic morphism which is not separable. Suppose otherwise. Then we would have

$$F\left(\{P_i\}\right) \overset{d}{\sim} \varphi_1 \oplus \varphi_2,$$

where both φ_1 and φ_2 must be umbilical quadratic harmonic morphisms with positive eigenvalue 1 since $F\left(\{P_i\}\right)$ is of this kind. It follows from [30] (Theorem 3.3) that there exist $\{Q_i\} \in C(2k,n)$ and $\{R_i\} \in C(2l,n)$ such that $F\left(\{Q_i\}\right) \overset{d}{\sim} \varphi_1$ and $F\left(\{R_i\}\right) \overset{d}{\sim} \varphi_2$. But then we would have

$$F\left(\{P_i\}\right) \overset{d}{\sim} \varphi_1 \oplus \varphi_2 \overset{d}{\sim} F\left(\{Q_i\}\right) \oplus F\left(\{R_i\}\right) \overset{d}{\sim} F\left(\{Q_i \oplus R_i\}\right).$$

This means that $\{P_i\} \overset{a}{\sim} \{Q_i \oplus R_i\}$, which is impossible since $\{P_i\}$ is assumed to be irreducible. On the other hand, it is obviously true that if $F(\{P_i\})$ is domain-equivalent to an unseparable umbilical quadratic harmonic morphism then $\{P_i\}$ is irreducible. Thus we obtain (iii), which completes the proof of the theorem. $\square$

3. O-systems and Clifford systems

Definition 3.1. *i) An (m,n)-dimensional* **O-system** *is an n-tuple $(\tau_1, \dots, \tau_n)$, denoted by $\{\tau_i\}$ for short, of orthogonal endomorphisms of $\mathbb{R}^m$ satisfying*

$$\tau_i^t \tau_j + \tau_j^t \tau_i = 2\delta_{ij} Id \ \ (i,j = 1,...,n). \tag{12}$$

The set of all (m,n)-dimensional O-systems is denoted by $O(m,n)$.
ii) A **representation** *of an O-system $\{\tau_i\} \in O(m,n)$ is a n-tuple $(a_1, \dots, a_n)$ of orthogonal matrices such that, with respect to some orthonormal basis in $\mathbb{R}^m$, a_i is the representation of τ_i for $i = 1,...,n$ respectively. An O-system is sometimes specified by its representation as $\{a_i\}$.*
iii) Let $\{\rho_i\} \in O(m,n)$ and $\{\tau_i\} \in O(l,n)$, then $\{\rho_i \oplus \tau_i\}$ is an O-system on $\mathbb{R}^{m+l}$, the so-called **direct sum** *of $\{\rho_i\}$ and $\{\tau_i\}$.*
iv) An O-system $\{\rho_i\} \in O(m,n)$ is said to be **irreducible** *if it is not possible to write $\mathbb{R}^m$ as a direct sum of two non-trivial subspaces which are invariant under all ρ_i.*
v) Two O-systems $\{\rho_i\}$, $\{\tau_i\} \in O(m,n)$ are said to be **algebraically equivalent**, *denoted by $\{\rho_i\} \overset{a}{\sim} \{\tau_i\}$, if there exists $\theta \in O(\mathbb{R}^m)$ such that $\tau_i = \theta \rho_i \theta^t$ for all $i = 1, \dots, n$. They are said to be* **geometrically equivalent**, *denoted by $\{\rho_i\} \overset{g}{\sim} \{\tau_i\}$, if $\{\varrho(\rho_i)\}$ and $\{\tau_i\}$ are algebraically equivalent for some $\varrho \in O(span\{\rho_1, \dots, \rho_n\}) \subset O(O(\mathbb{R}^m))$, where $O(\mathbb{R}^m)$ is provided with the inner product $\langle \tau, \rho \rangle = \frac{1}{m} tr(\rho^t \tau)$ for any $\tau, \rho \in O(\mathbb{R}^m)$.*

Remark 3.2. *We remark that n-tuples $\{\tau_i\}$ of orthogonal endomorphisms satisfying*

$$\tau_i\tau_j + \tau_j\tau_i = -2\delta_{ij}Id\ (i, j = 1, ..., n).$$

and n-tuples $\{a_k\}$ of skew symmetric endomorphisms satisfying Equation (12) have been used (see e.g., [22, 8, 31, 15]*) to study the representations of Clifford algebras. For examples, we know (see Lemma 24 in* [31]*) that there exists a bijective correspondence between the set of equivalence classes of $(n-1)$-tuples $\{a_k\}$ of skew symmetric endomorphisms of $\mathbb{R}^m$ satisfying Equation (12) and the set of orthogonal equivalence classes of representations $\chi(C_{n-1}, *)$ of Clifford algebra C_{n-1}. On the other hand, there is a classical result of Radon, Hurwitz and Eckmann's (see e.g.,* [8]*) saying that there exists $(\sigma(m)-1)$-tuples of* **skew symmetric and orthogonal** *endomorphisms of $\mathbb{R}^m$ satisfying Equation (12). In contrast, our results (see Theorem 4.4) claims the existence of range-maximal $(m, \sigma(m))$-dimensional O-systems which means that there exist! $\sigma(m)$-tuples of* **orthogonal** *endomorphisms of $\mathbb{R}^m$ satisfying Equation (12).*

In comparing with Ozeki and Takeuchi's result, we can use Definition (3.1) and the relation between O-systems and Clifford systems to establish the following

Proposition 3.3. *There is a bijective correspondence between the set of algebraic equivalence classes of $O(m, n)$ and that of orthogonal equivalence classes of representations $\chi(C_{n-1}, *)$ of Clifford algebra C_{n-1}.*

Lemma 3.4. 1) *Let $\{A_\alpha\}$ and $\{B_\alpha\}$ be representations of $\{P_\alpha\}$ and $\{Q_\alpha\} \in C(2m, n+1)$ respectively. Then the direct sum $\{P_\alpha \oplus Q_\alpha\}$ has a representation of the form*

$$\left\{ \begin{pmatrix} A_\alpha & 0 \\ 0 & B_\alpha \end{pmatrix} \right\}. \tag{13}$$

2) *Let $\{a_i\}$ and $\{b_i\}$ be representations of $\{\tau_i\}$ and $\{\rho_i\} \in O(m, n)$ respectively. Then the direct sum $\{\tau_i \oplus \rho_i\}$ has a representation of the*

form

$$\left\{\begin{pmatrix} a_i & 0 \\ 0 & b_i \end{pmatrix}\right\}. \tag{14}$$

3) $\{P_\alpha\} \in C(2m, n+1)$ (*respectively,* $\{\tau_i\} \in O(m,n)$) *is reducible if and only if it has a non-trivial representation of the form* (13) (*respectively,* (14)).

Proof. The proof of the lemma is trivial and is omitted. □

It can be checked that any Clifford system $\{P_\alpha\} \in C(2m, n+1)$ is algebraically equivalent to one given by Equation (8) with an associated n-tuple $\{\tau_i\}$ of orthogonal endomorphisms satisfying (9). Thus every Clifford system $\{P_\alpha\} \in C(2m, n+1)$ corresponds to an O-system $\{\tau_i\} \in O(m,n)$, and hence we have a surjective map

$$f : C(2m, n+1) \longrightarrow O(m,n), \tag{15}$$

with $f(\{P_\alpha\}) = \{\tau_i\}$.

Theorem 3.5. *Let f be the map defined by (15). Then*
(1) *f induces a bijective correspondence, $\overline{f} : C(2m, n+1)/ \overset{a}{\sim}\longrightarrow O(m,n)/ \overset{a}{\sim}$.*
(2) *f preserves the direct sum operations in the following sense:*

$$f(\{Q_\alpha \oplus R_\alpha\}) = f(\{Q_\alpha\}) \oplus f(\{R_\alpha\}), \tag{16}$$

and hence f preserves reducibility in the sense that $\{P_\alpha\}$ is irreducible if and only if $f(\{P_\alpha\})$ is irreducible.

Proof. For (1), it is evident that f induces an onto map

$$\overline{f} : C(2m, n+1)/ \overset{a}{\sim}\longrightarrow O(m,n)/ \overset{a}{\sim} .$$

It remains to prove that $\overline{f}$ is $1-1$. To this end, suppose that

$$\overline{f}([\{P_\alpha\}]) = [\{\tau_i\}] = \overline{f}([\{Q_\alpha\}]) = [\{\rho_i\}]$$

Then we have $\{\tau_i\} \overset{a}{\sim} \{\rho_i\}$. Thus there exists $\theta \in O(\mathbb{R}^m)$ such that $\tau_i = \theta\rho_i\theta^t$ for all $i = 1, \ldots, n$. Now one can check that

$$A = \begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix} \in O(\mathbb{R}^{2m})$$

and that

$$A \begin{pmatrix} 0 & \tau_i \\ \tau_i^t & 0 \end{pmatrix} A^t = \begin{pmatrix} 0 & \rho_i \\ \rho_i^t & 0 \end{pmatrix},$$

which means that $\{P_\alpha\}$ and $\{Q_\alpha\}$ are algebraically equivalent, and hence $[\{P_\alpha\}] = [\{Q_\alpha\}]$.

For the first statement of (2), we can check that

$$\{Q_\alpha \oplus R_\alpha\} \overset{a}{\sim} \left\{ \begin{pmatrix} I_m & 0 & 0 & 0 \\ 0 & I_l & 0 & 0 \\ 0 & 0 & -I_m & 0 \\ 0 & 0 & 0 & -I_l \end{pmatrix}, \begin{pmatrix} 0 & 0 & \tau_i & 0 \\ 0 & 0 & 0 & \rho_i \\ \tau_i^t & 0 & 0 & 0 \\ 0 & \rho_i^t & 0 & 0 \end{pmatrix} \right\}.$$

Thus, by definition of f and (1), we obtain Equation (16), which together with (1) gives the second statement of (2). This completes the proof of the theorem. □

Remark 3.6. *Combining* (i) *of Theorem 2.6 and* (1) *of Theorem 3.5, we have the following* $1-1$ *correspondences:*

$$O(m,n)/\overset{a}{\sim} \xrightarrow{1-1} C(2m,n+1)/\overset{a}{\sim} \xrightarrow{1-1} H_2^1(2m,n+1)/\overset{d}{\sim}.$$

Proposition 3.7. *Let* $\{\tau_i\} \in O(m,n)$ *be an O-system. Then*
(a) $\{\tau_i^t\}$ *is also an O-system in* $O(m,n)$.
(b) *any subset consisting of* k *elements of* $\{\tau_i\}$ *forms an* (m,k)*-dimensional O-system.*

Proof. The proof is a straightforward checking of the defining Equation (12) and is omitted. □

Remark 3.8. *It follows from Proposition 3.7 that if* $O(m,n) = \emptyset$ *then* $O(m,n+p) = \emptyset$ *for* $p \geq 1$. *On the other hand, by the direct sum operation, if* $O(m,n) \neq \emptyset$ *then* $O(km,n) \neq \emptyset$ *for* $k \geq 2$. *Therefore it is meaningful to put the qualifiers* **"range-maximal"** *and* **"domain-minimal"** *before a Clifford system and an O-system in a similar way as they are used in Definition 2.3.*

Proposition 3.9. *For $n \geq 2, k \in \mathbb{N}$, $O(2k+1,n)=\emptyset$. Geometrically, there exists no Q-nonsingular umbilical quadratic harmonic morphisms $\varphi : \mathbb{R}^{4k+2} \longrightarrow \mathbb{R}^{n+1}$ for $n \geq 2$.*

Proof. By Remark 3.8, we need only to show that $O(2k+1,2)=\emptyset$. In fact, if there were $\{\tau_1,\tau_2\} \in O(2k+1,2)$, then we would have $\tau_1,\tau_2 \in O(\mathbb{R}^{2k+1})$ satisfying $\tau_1^t\tau_2 = -\tau_2^t\tau_1$. It follows that

$$\det\tau_1 \det\tau_2 = (-1)^{2k+1}\det\tau_2\det\tau_1 = -\det\tau_1\det\tau_2,$$

and hence $\det\tau_1 = 0$ or $\det\tau_2 = 0$ which contradict the fact that τ_1,τ_2 are orthogonal. $\square$

4. O-systems and Orthogonal multiplications

Definition 4.1. *An* **orthogonal multiplication** *is an $\mathbb{R}$-bilinear map $\mu : \mathbb{R}^p \times \mathbb{R}^q \longrightarrow \mathbb{R}^n$ with $\|\mu(x,y)\| = \|x\|\|y\|$ for all $x \in \mathbb{R}^p, y \in \mathbb{R}^q$. We use $F(p,q;n)$ to denote the set of all orthogonal multiplications $\mu : \mathbb{R}^p \times \mathbb{R}^q \longrightarrow \mathbb{R}^n$.*

Though the existence of orthogonal multiplications is a purely algebraic problem it is closely related to the existence of interesting geometric objects such as vector fields on spheres and harmonic maps into spheres:

Fact A:(see [22]) If there exists an orthogonal multiplication $\mu \in F(n,m;m)$ then there exist $(n-1)$ linearly independent vector fields on $\mathbb{S}^{m-1}$.

Fact B:(see [11]) For any orthogonal multiplication $\mu \in F(p,q;n)$, the restriction of μ provides a harmonic map $\mathbb{S}^{p-1} \times \mathbb{S}^{q-1} \longrightarrow \mathbb{S}^{n-1}$. If $p=q$, the Hopf construction

$$H(x,y) = \big(\|x\|^2 - \|y\|^2,\ 2\mu(x,y)\big)\ : \mathbb{R}^p \times \mathbb{R}^p \longrightarrow \mathbb{R}^{n+1}. \tag{17}$$

provides a harmonic map $\mathbb{S}^{2p-1} \longrightarrow \mathbb{S}^n$.

In relating to quadratic harmonic morphisms, Baird has proved ([1] Theorem 7.2.7) that an orthogonal multiplication $\mu \in F(p,q;n)$ is a harmonic morphism if and only if the dimensions $p=q=n$ and

$n = 1, 2, 4$, or 8.

Now we give a link between O-systems and orthogonal multiplications as

Theorem 4.2. *There exists a* $1-1$ *correspondence between* $O(m, n)$ *and* $F(n, m; m)$*. The correspondence is given by*

$$O(m, n) \ni \{\tau_i\} \longmapsto \mu_\tau : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^m,$$

with

$$\mu_\tau(x, y) = x^i \mu_\tau(e_i, y) = x^i \tau_i(y). \tag{18}$$

where $x = x^i e_i$ *and* $\{e_i\}$ *is the standard basis for* $\mathbb{R}^n$.

Proof. It is an elementary fact from linear algebra that there is a $1-1$ correspondence, given by (18), between the set of n-tuples $\{\tau_i\}$ of linear endomorphisms of $\mathbb{R}^m$ and that of bilinear maps $\mu : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$. It remains to show that $\{\tau_i\}$ is an O-system if and only if the corresponding μ_τ is an orthogonal multiplication. To this end, we first note the following two facts:

(I) A linear transformation $\tau : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ is orthogonal if and only if τ sends $\mathbb{S}^{m-1}$ into $\mathbb{S}^{m-1}$;

(II) A bilinear map $\mu : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ is an orthogonal multiplication if and only if μ sends $\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}$ into $\mathbb{S}^{m-1}$.

Now suppose that $\{\tau_i\} \in O(m, n)$. Then $x^i \tau_i$ is orthogonal for any $(x^1, \ldots, x^n) \in \mathbb{S}^{n-1}$ since

$$(x^i \tau_i)(x^i \tau_i)^t = \sum_{i=1}^{n} (x^i)^2 Id + \sum_{i \neq j} x^i x^j (\tau_i^t \tau_j + \tau_j^t \tau_i) = Id. \tag{19}$$

Therefore, for any $x^i e_i \in \mathbb{S}^{n-1}$ and $y \in \mathbb{S}^{m-1}$ we have

$$\|\mu_\tau(x, y)\| = \|x^i \tau_i(y)\| = \|(x^i \tau_i)(y)\| = 1. \tag{20}$$

From this and (II) it follows that μ_τ is an orthogonal multiplication. Conversely, if μ_τ is an orthogonal multiplication then by (II), Equation (20) holds for any $x^i e_i \in \mathbb{S}^{n-1}$ and $y \in \mathbb{S}^{m-1}$. Thus, by (I), $x^i \tau_i$ is

orthogonal for any $(x^1,\dots,x^n)\in\mathbb{S}^{n-1}$. It follows that Equation(19) holds for arbitrary $(x^1,\dots,x^n)\in\mathbb{S}^{n-1}$, which implies that $\tau_i^t\tau_j+\tau_j^t\tau_i=0$ for $i,j=1,\dots,n$ and $i\neq j$. Thus $\{\tau_i\}\in O(m,n)$, which ends the proof of the theorem. $\square$

Remark 4.3. *It is known (Smith* [34]*, see also Eells and Ratto* [13]*) that there is a* $1-1$ *correspondence between* $F(p,q;n)$ *and the set of totally geodesic embeddings of* $\mathbb{S}^{p-1}$ *into* $O_{n,q}$*, the Stiefel manifold of orthogonal q-frames in n-space with suitable normalization. In particular, there is a* $1-1$ *correspondence between* $F(n,m;m)$ *and geodesic* $(n-1)$*-spheres in* $O(\mathbb{R}^m)$*.*

Now we are ready to give the following existence theorem for Clifford systems and O-systems.

Theorem 4.4. *A) Let* $\sigma(m)=2^c+8d$ *for any* $m\in\mathbb{N}$*, uniquely written as* $m=(2r+1)2^{c+4d}$ $(r,d\geq 0,\ 1\leq c\leq 3)$*. Then there exist range-maximal* $(m,\sigma(m))-$*dimensional O-systems, and hence range-maximal* $(2m,\sigma(m)+1)$*-dimensional Clifford systems.*
B) For any $n\in\mathbb{N}$ *there exist domain-minimal* $(2m(n),n+1)$*-dimensional Clifford systems, and hence domain-minimal* $(m(n),n)$*-dimensional O-systems for and only for the* $(m(n),n)$ *values listed in Table 1.*

n	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	$\dots$	*n+8*
$m(n)$	*1*	*2*	*4*	*4*	*8*	*8*	*8*	*8*	$\dots$	*16 m(n)*

TABLE 1.

Proof. From Theorem 4.2 we know that there exists an (m,n)-dimensional O-system if and only if there exists an orthogonal multiplication $\mu\in F(n,m;m)$. Now the existence of range-maximal O-systems follows from a classical result of Hurwitz [21] (see also Radon [33] and Eckmann [9]) that for any $m\in\mathbb{N}$, uniquely written as $m=(2r+1)2^{c+4d}$ $(r,d\geq 0,\ 1\leq c\leq 3)$, there exist orthogonal multiplications $\mu\in F(n,m;m)$ for $n=\sigma(m)=2^c+8d$ which is also the largest

number possible for such orthogonal multiplications to exist. The existence of range-maximal Clifford systems then follows from Theorem 3.5. Thus we obtain A).

For B), we note that a Clifford system or an O-system is domain-minimal if and only if it is irreducible. It follows from [15] that $(2m(n), n+1)$-dimensional irreducible Clifford systems exist precisely for the values of $(m(n), n)$ listed in Table 1. Again the relation between Clifford systems and O-systems (Theorem 3.5) gives the existence of domain-minimal O-systems, which completes the proof of the theorem. □

Theorem 4.5. (*Existence of umbilical quadratic harmonic morphisms*)
(a) *Let* $\sigma(m) = 2^c + 8d$ *for any* $m \in \mathbb{N}$, *uniquely written as* $m = (2r+1)2^{c+4d}$ $(r, d \geq 0,\ 1 \leq c \leq 3)$. *Then there exist range-maximal* Q*-nonsingular umbilical quadratic harmonic morphisms* $\varphi : \mathbb{R}^{2m} \longrightarrow \mathbb{R}^{\sigma(m)+1}$.
(b) *For any* $n \in \mathbb{N}$ *there exist domain-minimal* Q*-nonsingular umbilical quadratic harmonic morphisms* $\varphi : \mathbb{R}^{2m(n)} \longrightarrow \mathbb{R}^{n+1}$ *for and only for the* $(m(n), n)$ *values listed in Table 1.*
(c) *Any other* Q*-nonsingular umbilical quadratic harmonic morphisms into* $\mathbb{R}^{n+1}$ *exist precisely in the cases*

$$\mathbb{R}^{2km(n)} \longrightarrow \mathbb{R}^{n+1},\ k \geq 2,$$

where they are domain-equivalent to a direct sum of some domain-minimal umbilical quadratic harmonic morphism $\varphi : \mathbb{R}^{2m(n)} \longrightarrow \mathbb{R}^{n+1}$.

Proof. Using Theorem 4.4 and the map (10) we obtain statements (a) and (b) immediately. For (c) we first note from [30] that any Q-nonsingular umbilical quadratic harmonic morphism φ is domain-equivalent to $\lambda\varphi_0$ for φ_0 given by a Clifford system $\{P_i\} \in C(2km(n), n+1)$. It is known (see e.g. [15]) that any Clifford system is algebraically equivalent to a direct sum of irreducible ones. Thus, according to Table 1, any Clifford system $\{P_i\} \in C(2km(n), n+1)$ is algebraically equivalent to

$$\{P_i^1 \oplus \ldots \oplus P_i^k\},$$

where $\{P_i^\alpha\} \in C(2m(n), n+1)$ is irreducible for all $\alpha = 1, \ldots, k$. By using Theorem 2.6 we see that

$$\varphi_0 \stackrel{d}{\sim} F\left(\{P_i^1\}\right) \oplus \ldots \oplus F\left(\{P_i^k\}\right),$$

which gives (c) and hence we obtain the theorem. $\square$

As an immediate consequence, we have

Corollary 4.6. *Let $\varphi : \mathbb{R}^{2m} \longrightarrow \mathbb{R}^n$ be a Q-nonsingular umbilical quadratic harmonic morphism . Then either*
(a) φ is domain-minimal, in which case, $\varphi \stackrel{d}{\sim} \lambda\varphi_0$ for $\varphi_0 \in H_2^1(2m, n)$ given by an irreducible Clifford system, or
(b) $\varphi \stackrel{d}{\sim} \lambda(\varphi_1 \oplus \ldots \oplus \varphi_k)$, where all $\varphi_i \in H_2^1(2l, n)$ $(kl = m)$ are domain-minimal.

Now we give a splitting lemma which will give the existence of general quadratic harmonic morphisms.

Lemma 4.7. (The Splitting Lemma) *Let $\varphi : \mathbb{R}^{2m} \longrightarrow \mathbb{R}^n$ be a Q-nonsingular quadratic harmonic morphism . Then either*
(i) φ is umbilical, or
(ii) φ is domain-equivalent to a direct sum of Q-nonsingular umbilical quadratic harmonic morphisms.

Proof. Let $\varphi : \mathbb{R}^{2m} \longrightarrow \mathbb{R}^n$ be a Q-nonsingular quadratic harmonic morphism . We will prove the lemma by checking the following two cases:
Case I: All the positive eigenvalues $\lambda_1, \ldots, \lambda_m$ are distinct.

Claim 1: For $n \geq 3$, there exists no Q-nonsingular quadratic harmonic morphism $\varphi : \mathbb{R}^{2m} \longrightarrow \mathbb{R}^n$ with all positive eigenvalues distinct.
Proof of Claim 1: Since the composition of a quadratic harmonic morphism with distinct positive eigenvalues followed by a projection is again a quadratic harmonic morphism of this kind, it is enough to do the proof for the case $\mathbb{R}^{2m} \longrightarrow \mathbb{R}^3$. Suppose otherwise, if there were a

quadratic harmonic morphism of this kind, then by the Classification Theorem, we would have

(21)

$$D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_m \end{pmatrix}, \quad B_1 = \left(b^1_{ij}\right), \ B_2 = \left(b^2_{ij}\right) \in GL(\mathbb{R}, m)$$

and satisfy Equation (6). Now from the first equation of (6) we know that B_i must be of diagonal form

$$B_i = \begin{pmatrix} b^i_{11} & 0 & \dots & 0 \\ 0 & b^i_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b^i_{mm} \end{pmatrix}.$$

But then the third equation of (6) says that $b^1_{jj} b^2_{jj} = 0$ for $j = 1, \dots, m$. This implies that either $b^1_{jj} = 0$ or $b^2_{jj} = 0$, which is impossible since B_i $(i = 1, 2)$ is non-singular. This ends the proof of Claim 1..

Claim 2: Any Q-nonsingular quadratic harmonic morphism $\varphi : \mathbb{R}^{2m} \longrightarrow \mathbb{R}^2$ with all positive eigenvalues distinct is domain-equivalent to a direct sum of umbilical ones.

Proof of Claim 2: We know from [30] that any Q-nonsingular quadratic harmonic morphism $\varphi : \mathbb{R}^{2m} \longrightarrow \mathbb{R}^2$ is domain-equivalent to the normal form

$$\tilde{\varphi}(X) = \left(X^t \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix} X, \ X^t \begin{pmatrix} 0 & B_1 \\ B_1^t & 0 \end{pmatrix} X \right), \tag{22}$$

where D and B_1 are as in (21) and satisfy Equation (6). From Equation (6) and the hypothesis on $\lambda_i' s$ we deduce that B_1 must be diagonal form

$$B_1 = \begin{pmatrix} \pm\lambda_1 & 0 & \dots & 0 \\ 0 & \pm\lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \pm\lambda_m \end{pmatrix}.$$

Inserting this into (22), we have

$$\begin{aligned} \tilde{\varphi}(X) = (&\lambda_1(x_1^2 - x_{m+1}^2) + \dots + \lambda_m(x_m^2 - x_{2m}^2), \\ &\pm 2\lambda_1 x_1 x_{m+1} \pm \dots \pm 2\lambda_m x_m x_{2m}). \end{aligned}$$

It is easy to check that

$$\varphi \overset{d}{\sim} \tilde{\varphi} \overset{d}{\sim} \lambda_1 \varphi_0 \oplus \ldots \oplus \lambda_m \varphi_0,$$

where $\varphi_0 : \mathbb{R}^2 \cong \mathbb{C} \longrightarrow \mathbb{R}^2 \cong \mathbb{C}$ with $\varphi_0(z) = z^2$ which is clearly umbilical. Thus we obtain Claim 2. Combining Claim 1 and 2 we see that the lemma is true for Case I.

Case II: φ has some equal positive eigenvalues.

Without loss of generality, we may assume that the positive eigenvalues satisfy $\lambda_1 = \ldots = \lambda_k \neq \lambda_l$ $(k < l \le m)$. By the Classification Theorem, we know that $\varphi \overset{d}{\sim} \tilde{\varphi}$ given by the normal form (5) with B_i $(i = 1, \ldots, n-1)$, $D \in GL(\mathbb{R}, m)$ satisfying (6). Now using Equation (6) and the hypothesis on λ_i's we can check that B_i must take the form

$$B_i = \begin{pmatrix} b_i & 0 \\ 0 & c_i \end{pmatrix}, \ i = 1, \ldots, n-1,$$

where $b_i \in GL(\mathbb{R}, k)$, $c_i \in GL(\mathbb{R}, m-k)$. Writing
$(\underbrace{x_1, \ldots, x_k}, \underbrace{x_{k+1}, \ldots, x_m}, \underbrace{x_{m+1}, \ldots, x_{m+k}}, \underbrace{x_{m+k+1}, \ldots, x_{2m}})$
as (X_1, X_2, X_3, X_4), we can check that after an orthogonal change of the coordinates of the form

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} = \begin{pmatrix} I_k & 0 & 0 & 0 \\ 0 & 0 & I_{m-k} & 0 \\ 0 & I_k & 0 & 0 \\ 0 & 0 & 0 & I_{m-k} \end{pmatrix} \begin{pmatrix} \tilde{X}_1 \\ \tilde{X}_2 \\ \tilde{X}_3 \\ \tilde{X}_4 \end{pmatrix},$$

we have, $\tilde{\varphi}(\tilde{X}) = \varphi_1(\tilde{X}_1, \tilde{X}_2) + \overline{\varphi}_2(\tilde{X}_3, \tilde{X}_4)$, i.e., $\tilde{\varphi} \overset{d}{\sim} \varphi_1 \oplus \overline{\varphi}_2$ with $\varphi_1 : \mathbb{R}^{2k} \longrightarrow \mathbb{R}^n$ and $\overline{\varphi}_2 : \mathbb{R}^{2(m-k)} \longrightarrow \mathbb{R}^n$ both quadratic maps. By using Theorem 1.5 in [30] and the fact that $\tilde{\varphi}(\tilde{X})$ is a Q-nonsingular quadratic harmonic morphism we can prove that both φ_1 and $\overline{\varphi}_2$ are Q-nonsingular quadratic harmonic morphisms with φ_1 umbilical since $\lambda_1, \ldots, \lambda_k$ are supposed to be equal. Now the same process applies to $\overline{\varphi}_2$ and so on until we obtain

$$\varphi \overset{d}{\sim} \tilde{\varphi} \overset{d}{\sim} \varphi_1 \oplus \ldots \oplus \varphi_i,$$

for all umbilical $\varphi_1, \dots, \varphi_i$.
Therefore, we have seen from Case I and Case II that either φ is umbilical or, φ is domain-equivalent to a direct sum of Q-nonsingular umbilical ones. This ends the proof of the lemma. $\square$

Recall that a domain-minimal quadratic harmonic morphism is always Q-nonsingular and not separable, we see immediately, from the Splitting Lemma, the following

Corollary 4.8. *Any domain-minimal quadratic harmonic morphism $\varphi : \mathbb{R}^{2m} \longrightarrow \mathbb{R}^n$ is umbilical.*

From the Splitting Lemma and Theorem 4.5 we obtain the following existence theorem for general quadratic harmonic morphisms.

Theorem 4.9. *For $n \in \mathbb{N}$, Q-nonsingular quadratic harmonic morphisms exist precisely in the cases $\mathbb{R}^{2km(n)} \longrightarrow \mathbb{R}^{n+1}$ for $k \in \mathbb{N}$, and $m(n)$ depending on n given by Table 1. Furthermore*
(I) If $k = 1$, then φ is domain-minimal and hence umbilical, and $\varphi \overset{d}{\sim} \lambda\varphi_0$ for $\varphi_0 \in H_2^1(2m(n), n+1)$. Otherwise
(II) φ is domain-equivalent to a direct sum of k domain-minimal quadratic harmonic morphisms, i.e.,

$$\varphi \overset{d}{\sim} \lambda_1\varphi_1 \oplus \dots \oplus \lambda_k\varphi_k, \tag{23}$$

for $\varphi_i \in H_2^1(2m(n), n+1)$, given by irreducible Clifford systems.

Remark 4.10. *Note that $\lambda_1, \dots, \lambda_k$ in (23) are the distinct positive eigenvalues of φ. If we allow some but not all of them to be zero then our results include also Q-singular cases. Thus Equation (23) gives, up to domain-equivalence, a general form of quadratic harmonic morphisms $\mathbb{R}^{2km(n)} \longrightarrow \mathbb{R}^{n+1}$.*

Corollary 4.11. *Let $\varphi : \mathbb{R}^{2km(n)} \longrightarrow \mathbb{R}^{n+1}$ be a quadratic harmonic morphism . Then*
i) for $n \not\equiv 0 \mod 4$, $\varphi \overset{d}{\sim} \lambda_1\varphi_0 \oplus \dots \oplus \lambda_k\varphi_0$, where $\varphi_0 : \mathbb{R}^{2m(n)} \longrightarrow \mathbb{R}^{n+1}$ is a domain-minimal quadratic harmonic morphism given by an

irreducible Clifford system. Otherwise
ii) $n \equiv 0 \mod 4$, *φ belongs to one of* 2^{k-1} *bi-equivalent classes.*

Proof. From [15], we know that there exists only one algebraic equivalent class in $C(2m(n), n+1)$ for $n \not\equiv 0 \mod 4$, and two for $n \equiv 0 \mod 4$. The the statement *i)* now follows from Theorem 4.9 and Remark 4.10.. For statement *ii)*, we first note (see [15]) that two algebraically different irreducible Clifford systems differ only by a minus sign before one, say the last, of their elements. Correspondingly, two domain-minimal quadratic harmonic morphisms $\lambda\varphi_1$ and $\lambda\varphi_2$ from two different domain-equivalent classes differ only by a minus sign before, say, the last component functions. Therefore, in constructing direct sum of the form $\lambda_1\varphi_1 \oplus \ldots \oplus \lambda_k\varphi_k$ with fixed tuple $(\lambda_1, \ldots, \lambda_k)$, we get 2^k possibilities, of which, half can be obtained from the other by an orthogonal change of the coordinates in the range space as one can check.! Thus *ii)* follows, which completes the proof of the corollary. □

5. Properties of quadratic harmonic morphisms

Definition 5.1. *Let* (M, g) *be a space form (i.e., either Euclidean sphere* $\mathbb{S}^m$*, Euclidean space* $\mathbb{R}^m$ *or hyperbolic space* $\mathbb{H}^m$*). A smooth function* $f : M \longrightarrow \mathbb{R}$ *is called* **isoparametric** *if*

$$\|d\, f(x)\|^2 = \psi_1(f(x)),$$
$$\triangle f(x) = \psi_2(f(x)),$$

for some smooth functions $\psi_1,\ \psi_2 : \mathbb{R} \longrightarrow \mathbb{R}$.

Such functions were introduced by Cartan [7] in 1938. Their description on Euclidean space and hyperbolic space is relatively trivial, but on the sphere they are rich in geometry. More recent studies on such functions have been made in [15, 25, 26, 31, 32, 35, 36].

It is well-known that all isoparametric functions $f : \mathbb{S}^{m-1} \longrightarrow \mathbb{R}$ arise from the restriction of a homogeneous polynomial $F : \mathbb{R}^m \longrightarrow \mathbb{R}$

of degree p with

$$\| \bigtriangledown F\|^2 = p^2\|x\|^{2p-2}, \tag{24}$$

$$\triangle F = c\|x\|^{p-2}. \tag{25}$$

where $c = 0$ if the multiplicities of the distinct principal curvatures are equal.

Suppose we are given a homogeneous polynomial $F : \mathbb{R}^m \longrightarrow \mathbb{R}$ of degree p with $c = 0$, satisfying (24) and (25). Given any $A \in O(\mathbb{R}^m)$, we define $G : \mathbb{R}^m \longrightarrow \mathbb{R}$ by putting $G = F \circ A$. Then G is also a polynomial satisfying (24) and (25). Baird noted that if $A \in O(\mathbb{R}^m)$ can be so chosen that $\langle \bigtriangledown G_X, \bigtriangledown F_X \rangle = 0$ for any $X \in \mathbb{R}^m$, then $\varphi = (F, G)$ gives a nontrivial harmonic morphism defined by homogeneous polynomials of degree p. This provides a possible way to construct polynomial harmonic morphisms from a single homogeneous polynomial. However, this method fails in a case of homogeneous degree 4 polynomial harmonic morphism $\mathbb{R}^6 \longrightarrow \mathbb{R}^2$ (see Theorem 8.3.5 in [1]). We shall prove that this method works for any quadratic harmonic morphisms.

Theorem 5.2. *Any quadratic harmonic morphism* $\varphi : \mathbb{R}^{2km(n)} \longrightarrow \mathbb{R}^{n+1}$ *arises from a single quadratic function* $F : \mathbb{R}^{2km(n)} \longrightarrow \mathbb{R}$, $F = \lambda_1 F_0 \oplus \ldots \oplus \lambda_k F_0$, *where* $\lambda_i \geq 0$ *are constants with at least one not zero, and* $F_0 : \mathbb{R}^{2m(n)} \longrightarrow \mathbb{R}$,

$$F_0(x_1, \ldots, x_{2m}) = {x_1}^2 + \ldots + {x_m}^2 - {x_{m+1}}^2 - \ldots - x_{2m}^2.$$

That is, if we write $\varphi = (\varphi^1, \ldots, \varphi^{n+1})$, *then*
$\varphi^1 = F$, $\varphi^i = F \circ G_i$ $(i = 2, \ldots, n+1)$ *for* $G_i \in O(\mathbb{R}^{2km(n)})$.

Proof. Note that if $G_1, \ldots, G_k \in O(\mathbb{R}^{2m(n)})$ then $G_1 \oplus \ldots \oplus G_k \in O(\mathbb{R}^{2km(n)})$. On the other hand, we have seen from Theorem 4.9 that a quadratic harmonic morphism exists precisely in the case $\varphi : \mathbb{R}^{2km(n)} \longrightarrow \mathbb{R}^{n+1}$, where we have

$$\varphi \overset{d}{\sim} \lambda_1\varphi_1 \oplus \ldots \oplus \lambda_k\varphi_k,$$

for domain-minimal $\varphi_i \in H_2^1(2m(n), n{+}1)$, given by irreducible Clifford systems. Thus it suffices to show the following

Claim: Any domain-minimal $\varphi_i \in H_2^1(2m(n), n+1)$ arises from a single quadratic function F_0.

Proof of Claim: It follows from Corollary 4.8 and the Classification Theorem that

$$\varphi_i \overset{d}{\sim} \left(X^t \begin{pmatrix} I_m & 0 \\ 0 & -I_m \end{pmatrix} X,\ X^t A_1 X, \dots, X^t A_n X \right).$$

where the component matrices A_i, by the Rank Lemma in [30], have the same rank, the same index and the same spectrum as $\begin{pmatrix} I_m & 0 \\ 0 & -I_m \end{pmatrix}$ does. Therefore from the theory of real quadratic forms we know that there exist $G_i^\alpha \in O(\mathbb{R}^{2m(n)})$ such that $\varphi_i^\alpha = \varphi_i^1 \circ G_i^\alpha = F_0 \circ G_i^\alpha$ $(\alpha = 2, \dots, n+1)$, which ends the proof of the claim. $\square$

Example 5.3. *We can check that* $\varphi : \mathbb{R}^8 \longrightarrow \mathbb{R}^3$ *given by*

$$\begin{aligned} \varphi =& (2x_1^2 + 2x_2^2 + 3x_3^2 + 3x_4^2 - 2x_5^2 - 2x_6^2 - 3x_7^2 - 3x_8^2, \\ & 4x_1x_5 + 4x_2x_6 + 6x_3x_8 - 6x_4x_7, \\ & -4x_1x_6 + 4x_2x_5 + 6x_3x_7 + 6x_4x_8) \end{aligned}$$

is a quadratic harmonic morphism. Let $F = 2F_0 \oplus 3F_0$ *with*

$$F_0 : \mathbb{R}^4 \longrightarrow \mathbb{R}, \quad F_0(x_1, \dots, x_4) = x_1^2 + x_2^2 - x_3^2 - x_4^2.$$

We can further check that φ *arises from* F *since*

$$\begin{aligned} &\varphi^1 = F, \quad \varphi^2 = F \circ (G_1 \oplus G_2) = F \circ G^1, \\ &\varphi^3 = F \circ (G_2 \oplus G_3) = F \circ G^2. \end{aligned}$$

for

$$G_1 = \begin{pmatrix} \frac{1}{\sqrt2} & 0 & \frac{1}{\sqrt2} & 0 \\ 0 & \frac{1}{\sqrt2} & 0 & \frac{1}{\sqrt2} \\ -\frac{1}{\sqrt2} & 0 & \frac{1}{\sqrt2} & 0 \\ 0 & -\frac{1}{\sqrt2} & 0 & \frac{1}{\sqrt2} \end{pmatrix}, \ G_2 = \begin{pmatrix} \frac{1}{\sqrt2} & 0 & 0 & -\frac{1}{\sqrt2} \\ 0 & \frac{1}{\sqrt2} & \frac{1}{\sqrt2} & 0 \\ 0 & -\frac{1}{\sqrt2} & \frac{1}{\sqrt2} & 0 \\ \frac{1}{\sqrt2} & 0 & 0 & \frac{1}{\sqrt2} \end{pmatrix},$$

$$G_3 = \begin{pmatrix} \frac{1}{\sqrt2} & 0 & 0\frac{1}{\sqrt2} & \\ 0 & \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} & 0 \\ 0 & \frac{1}{\sqrt2} & \frac{1}{\sqrt2} & 0 \\ -\frac{1}{\sqrt2} & 0 & 0 & \frac{1}{\sqrt2} \end{pmatrix} \in O(\mathbb{R}^4).$$

Ferus, Karcher and Münzner have noted in [15] that some Clifford systems can be extended, by adding one member, to be a Clifford system with one more range-dimension. Our next theorem gives the conditions on the **range-extendability** of Clifford systems, O-systems and quadratic harmonic morphisms.

Theorem 5.4. *Any domain-minimal quadratic harmonic morphisms (respectively, Clifford systems, or O-systems) which are not range-maximal can be extended, by adding component functions (respectively, system members), to be a range-maximal one.*

Proof. By Corollary 4.8, any domain-minimal quadratic harmonic morphism $\varphi : \mathbb{R}^{2m(n)} \longrightarrow \mathbb{R}^{n+1}$ is Q-nonsingular and umbilical, and hence it arises from a single quadratic function λF_0 for $F_0 : \mathbb{R}^{2m(n)} \longrightarrow \mathbb{R}$,

$$F_0(x_1, \ldots , x_{2m}) = {x_1}^2 + \ldots + {x_m}^2 - {x_{m+1}}^2 - \ldots - x_{2m}^2.$$

On the other hand, any domain-minimal and range-maximal quadratic harmonic morphism $\varphi : \mathbb{R}^{2m} \longrightarrow \mathbb{R}^{\sigma(m)+1}$ also arises from the quadratic function λF_0. Thus if $n \leq \sigma(m)$ we can add some component functions of the form $\lambda F_0 \circ G^{\alpha}$ untill we get a range-maximal one. This proves the result for quadratic harmonic morphisms. The corresponding results for Clifford systems and O-systems follow from the relationships (Theorems 2.6 and 3.5) between quadratic harmonic morphisms, Clifford systems and O-systems. □

Note that the quadratic harmonic morphism in Example 5.3 is not domain-minimal. Though it is not range-maximal either, it cannot be extended to be a range-maximal one as one can check easily. On the other hand, the standard multiplication of complex numbers is neither domain-minimal nor range-maximal. But, as we will see from the following remark that it can be extended to be a range-maximal one.

Remark 5.5. *Theorem 5.4 gives a method of constructing quadratic harmonic morphisms from some given ones. It is interesting to note*

that this construction includes the Hopf construction maps of the standard multiplication $p_n : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^n$, $(n = 1, 2, 4 \text{ or } 8)$ *of real, complex, quaternionic and Cayley numbers as special cases: Since* $2p_n : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^n$ *is also a quadratic harmonic morphism and for* $n = 4$ *or* 8 *they are domain-minimal but not range-maximal. Therefore, by Theorem 5.4, they can be extended, by adding one component function to be range-maximal one as* $H(x, y) = (\|x\|^2 - \|y\|^2,\ 2p_n(x, y))$, *which is exactly the Hopf construction map. The cases for* $n = 1$ *or* 2 *are easier to check. It can be further checked that in the above cases there are at most two possible ways of adding one component function in doing the extension.*

Theorem 5.6. *For* $n = 1,\ 4$ *or* 8, *any quadratic harmonic morphism* $\varphi \in H_2(2n, n)$ *is domain-equivalent to a constant multiple of the standard multiplications of the real algebras of real, quaternionic and Cayley numbers respectively.*

Proof. We first note that in all three cases in question, φ is domain-minimal and hence Q-nonsingular and umbilical. Therefore , by (a) of Corollary 4.6, $\varphi \overset{d}{\sim} \lambda\varphi_0$ for $\varphi_0 \in H_2^1(2n, n)$ given by an irreducible Clifford system $\{P_i\} \in C(2n, n)$ which, in all three cases, belongs to exactly one algebraically equivalent class (see [15]). On the other hand, it is known (see Baird [1] Theorem 7.2.7) that the standard multiplications of the real algebras of real, quaternionic and Cayley numbers are, respectively, in the class. Thus we obtain the theorem. □

Proposition 5.7. *For* $n = 1,\ 2,\ 4$ *or* 8, *any quadratic harmonic morphism* $\varphi \in H_2(2n, n+1)$ *is bi-equivalent to a constant multiple of the Hopf construction map in the corresponding cases, and therefore,* φ *restricts to harmonic morphism* $\mathbb{S}^{2n-1} \longrightarrow \mathbb{S}^n(\lambda)$, *where* $\mathbb{S}^n(\lambda)$ *denotes the Euclidean sphere of radius* λ.

Proof. $n = 1$ is trivial. For $n = 2$ the results have been obtained in [30], where all quadratic harmonic morphisms $\varphi \in H_2(4, 3)$ are determined explicitly and are proved to be domain-equivalent to a constant

multiple of the Hopf construction map. Now we give a proof which treats all four cases. Note that in all these cases, any quadratic harmonic morphism φ is domain-minimal (see Theorem 4.5) and hence Q-nonsingular and umbilical. Therefore , $\varphi \overset{d}{\sim} \lambda\varphi_0$, for φ_0 given by an irreducible Clifford system $\{P_i\} \in C(2n, n+1)$. Now the first half of the statement in the proposition follows from Proposition 2.6 and the fact (see [15]) that there exists exactly one geometric equivalence classes in these cases. The second half of the statement is trivial and is omitted. $\square$

Remark 5.8. *From Proposition 5.7 it follows that any quadratic harmonic morphism* $\varphi : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{n+1}$, $(n = 1,\ 2, 4$ *or* $8)$ *restricts to a composition of the classical Hopf fibration followed by a homothety. Thus, in a sense, Proposition 5.7 generalizes part of a result of Eells and Yiu (see* [14]*) which says that if* $\phi : \mathbb{S}^m \longrightarrow \mathbb{S}^n$ *is the restriction of a homogeneous polynomial harmonic morphism* $\Phi : \mathbb{R}^{m+1} \longrightarrow \mathbb{R}^{n+1}$. *Then* ϕ *is isometric to the classical Hopf fibration.*

Remark 5.9. *Concerning quadratic harmonic morphisms between spheres we know (see* [14]*) that a quadratic map* $\varphi : \mathbb{S}^m \longrightarrow \mathbb{S}^n$ *is a harmonic morphism if and only if it is isometric to one of the classical Hopf fibrations* $\mathbb{S}^{2n-1} \longrightarrow \mathbb{S}^n$ *for* $n = 1, 2, 4,$ *or* 8.

From Remark 3.6 we see that any umbilical $\varphi \in H_2^1(2m, n+1)$ is associated with an orthogonal multiplication $\mu_\varphi \in F(n, m; m)$ by

$$\varphi \mapsto \{P_\alpha\} \mapsto \{\tau_i\} \mapsto \mu_\varphi. \tag{26}$$

It is interesting to note that the Hopf construction maps of the standard multiplications of the real algebras of real, complex, quaternionic and Cayley numbers are the only umbilical quadratic harmonic morphisms which correspond to orthogonal multiplications that are also harmonic morphisms.

Proposition 5.10. *Let $\varphi \in H_2^1(2m, n+1)$, and $\mu_\varphi \in F(n, m; m)$ be the corresponding orthogonal multiplication via (26). Then μ_φ is a harmonic morphism if and only if φ is bi-equivalent to the Hopf construction maps of the standard multiplications of real, complex, quaternionic and Cayley numbers.*

Proof. From Baird [1] it follows that, an orthogonal multiplication $\mu_\varphi \in F(n, m; m)$ is a harmonic morphism if and only if $n = m = 1,\ 2,\ 4,\ or\ 8$. This implies that $\varphi \in H_2^1(2n, n+1)$ for $n = 1,\ 2,\ 4,\ or\ 8$ which, together with Proposition 5.7, gives the required results. □

It is easily checked that any quadratic harmonic morphism $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is domain-equivalent to $\varphi_0(z) = z^2 : \mathbb{R}^2 \cong \mathbb{C} \longrightarrow \mathbb{R}^2 \cong \mathbb{C}$. A proof similar to that of Proposition 5.7 can be applied to give the following

Theorem 5.11. *Any quadratic harmonic morphism $\varphi : \mathbb{R}^m \longrightarrow \mathbb{R}^2$ is domain-equivalent to*

$$\lambda_1 z_1^2 + \ldots + \lambda_k z_k^2 : \quad \mathbb{C}^k \longrightarrow \mathbb{C},$$

where $k = [\frac{m}{2}]$, and $\lambda_i \geq 0$ $(i = 1, \ldots, k)$ with at least one not zero.

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